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**235. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.**

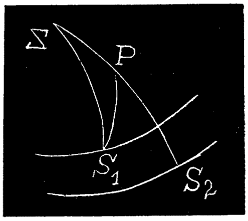
The latitude of a place and two circles parallel to the horizon being given, to determine the declination of a heavenly body whose apparent time of passage from one circle to the other shall be a minimum.

**I. Solution by the PROPOSER.**

Employing the usual notation, let $Z$ be the zenith, $P$ the celestial pole, $S_1$, $S_2$, the positions of a heavenly body on the parallel circles, the polar distances $PS_1$ and $PS_2$ being equal.

Let $\angle ZPS_1=\phi_1$, and $\angle ZPS_2=\phi_2$; and let $PS_1=PS_2=x$, arc $ZS_1=\alpha_1$, $ZS_2=\alpha_2$, latitude$=\lambda$, declination$=D$.

Then by the conditions of the problem the heavenly body is to pass along the arc $S_1S_2$ in the shortest possible time. In other words $\angle S_1PS_2$ is to be a minimum.



$$\therefore \frac{d}{dx}S_1PS_2=\frac{d}{dx}(\phi_2-\phi_1)=0,$$

$$\text{but } \frac{d\phi_1}{dx}=-\frac{\cot S_1}{\sin x} \text{ and } \frac{d\phi_2}{dx}=-\frac{\cot S_2}{\sin x}.$$

$$\therefore \frac{\cot S_1}{\sin x}=\frac{\cot S_2}{\sin x}. \quad \therefore S_1=S_2;$$

$$\text{but } \cos S_1=\frac{\sin\lambda-\cos\alpha_1\cos x}{\sin\alpha_1\sin x} \text{ and } \cos S_2=\frac{\sin\lambda-\cos\alpha_2\cos x}{\sin\alpha_2\sin x}.$$

$$\therefore \frac{\sin\lambda-\cos\alpha_1\cos x}{\sin\alpha_1}=\frac{\sin\lambda-\cos\alpha_2\cos x}{\sin\alpha_2}.$$

$$\text{Whence } \cos x=\frac{\cos\frac{1}{2}(\alpha_2+\alpha_1)}{\cos\frac{1}{2}(\alpha_2-\alpha_1)}.\sin\lambda.$$

The declination being the complement of the polar distance we have, therefore,

$$\sin D=\frac{\cos\frac{1}{2}(\alpha_2+\alpha_1)}{\cos\frac{1}{2}(\alpha_2-\alpha_1)}\sin\lambda.$$

In the special case where $\alpha_1=\frac{1}{2}\pi$, and $\alpha_2=\frac{1}{2}\pi+2\delta$, this expression is reduced to $\sin D=-\tan\delta\sin\lambda$. This formula may have important applications. Thus, take for instance, the case of the sun, and supposing $2\delta$ to be its angular depression below the horizon when twilight begins in the morning or ends in the evening, we can calculate the time of shortest twilight at a given place by means of the relation $-\sin D=\tan\delta\tan\lambda$, the negative sign indicating that when the latitude is north the declination will be south, and *vice versa.*

II. **Solution by W. D. LAMBERT, Washington, D. C.**

This is a slight generalization of Bernoulli's classic problem of the shortest twilight. The expressions that occur in the analytic solution are long and cumbrous, but present no essential difficulty. An outline of the analytic solution is given in Hutton's *Course of Mathematics*, Vol. II, page 385 (12th ed.). The following synthetic solution may be new to many readers.

Suppose for definiteness the body, $S$, to be west of the meridian. Let $P$ be the pole of the celestial sphere, $Z$ the zenith when the body is in the upper circle of zenith distance $z=ZS$. After the lapse of a period $t$, the point of the sphere once occupied by $Z$ has moved to $Z'$, and the body $S$ is now in $S'$ at a zenith distance $z+2a=ZS'$. Let $\phi$ denote the latitude, $=90^\circ-PZ$, and $\delta$ the declination $=90^\circ-PS=90^\circ-PS'$. $ZPZ'=SPS'=t$.

The arc $ZZ'$ is found from the isosceles triangle $ZPZ'$ and is given by $\sin\frac{1}{2}ZZ'=\cos\phi\,\sin\frac{1}{2}t$, so that $ZZ'$ increases with $t$, and will have a minimum value when $t$ has one. From the triangle $ZZ'S'$,

$$\cos ZZ'=\cos z\cos(z+2a)+\sin z\sin(z+2a)\cos ZS'Z' \ldots (1),$$

$z$ and $z+2a$ are given. The only variable on the right is $ZS'Z'$, and since $\sin z$ and $\sin(z+2a)$ are always positive, by increasing $\cos ZS'Z'$ we increase $\cos ZZ'$ and thereby diminish arc $ZZ'$. $ZZ'$, and with it $t$, will be a minimum when $ZS'Z'=0$, and then by (1), $ZZ'=2a$, which implies that $Z'$ falls on $S'Z$. From the triangle $ZPS'$,

$$\sin\delta=\cos(z+2a)\sin\phi+\cos\phi\sin(z+2a)\cos PZS' \ldots (2),$$

but from the triangle $ZPZ'$, $\cos PZS'=\tan a\tan\phi$, so that (2) may be reduced to

$$\sin\delta=\frac{\sin\phi\cos(z+a)}{\cos a} \ldots (3),$$

which is the required declination.

The corresponding time is given by

$$\sin\frac{t}{2}=\frac{\sin a}{\cos\phi} \ldots (4).$$

For the problem of shortest twilight, $Z=90^\circ$, and $2a$ is usually taken as 18°. Equations (2) and (3) then show that the central parts of this country, of latitude say 40°, have their shortest twilights when the sun is 5° 51′ south of the equator, or about March 6, and October 8. These twilights last 1 hour, 34 minutes.

It is easily seen from a figure that the parallactic angles at the beginning and end of the minimum period, namely $PSZ$ and $PS'Z$, are equal, and the azimuths $PZS$ and $PZS'$ are supplementary. Some such symmetry as this regard to the prime vertical might be anticipated, for the rate of descent in altitude of a a body is a maximum when on the prime vertical.

**Excellent solutions of this problem were received from G. B. M. Zerr, J. Scheffer, and G. W. Greenwood.**